

A new dynamical reflection algebra and related quantum integrable systems

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Abstract

We propose a new dynamical reflection algebra, distinct from the previous dynamical boundary algebra and semi-dynamical reflection algebra. The associated Yang-Baxter equations, coactions, fusions, and commuting traces are derived. Explicit examples are given and quantum integrable Hamiltonians are constructed. They exhibit features similar to the Ruijsenaars-Schneider Hamiltonians.

MSC: 81R12, 16T15, 16T25

June 2011
LAPTH-019/2011

1 Introduction

Quantum reflection algebras originated from the studies of consistent sufficient conditions on boundary processes required to preserve quantum integrability for systems with boundaries. Originating with the study of systems on a half-line [1], they were systematically developed by Sklyanin [2] (see also [3]) and more generally extended to the form of quadratic exchange algebra in [4]. The general structure is then:

$$A_{12} K_1 B_{12} K_2 = K_2 C_{12} K_1 D_{12}, \quad (1.1)$$

where K is the matrix encapsulating the generators of the quadratic algebra and A, B, C and D are \mathbb{C} -number structure matrices. Consistency of (1.1) with requirements that the quantum exchange algebra be associative is guaranteed by the Yang-Baxter equation for A and D , and adjoint Yang-Baxter for A, B and for C, D . In particular, A and D are thus identified with quantum R -matrices.

Here and throughout the paper, the indices $1, 2, \dots$ in $ABCD$ and K label the so-called "auxiliary spaces", i.e. a vector space \mathcal{V} on which the representation in (1.1) is defined. The space \mathcal{V} may be finite or infinite dimensional: this latter includes the case when \mathcal{V} is a loop space $V \otimes \mathbb{C}[[z]]$ and A, B, C, D and K then depend on (respectively) two or one complex spectral parameter(s). Explicit dependence in spectral parameters may be omitted to lighten notation, except when shifts (in the spectral parameter) occur: in that case, it will be explicitly written, as e.g. in section 3. Note however that the dynamical parameter \mathbf{q} may be explicit, even when the spectral parameter is omitted and attached to the auxiliary space index. Hence A_{12} stands for $A_{12}(\mathbf{q})$ that is itself understood as $A_{12}(z_1, z_2; \mathbf{q})$, while $A_{21} \equiv A_{21}(\mathbf{q}) \equiv A_{21}(z_2, z_1; \mathbf{q})$.

The possibility of extending the notion of reflection algebra to the case where A and D become dynamical quantum R -matrices arose first when defining integrable boundary conditions for IRF models [5]. The "boundary dynamical algebra" there defined reads:

$$A_{12}(\mathbf{q}) K_1(\mathbf{q} + h^{(2)}) B_{12}(\mathbf{q}) K_2(\mathbf{q} + h^{(1)}) = K_2(\mathbf{q} + h^{(1)}) C_{12}(\mathbf{q}) K_1(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}), \quad (1.2)$$

where \mathbf{q} encodes the dynamical parameters, interpreted as coordinates q_i , $i = 1, \dots, r$, on the dual \mathfrak{h}^* of the Cartan subalgebra \mathfrak{h} of some Lie algebra. $h_i^{(j)} \in \mathfrak{h}$, $i = 1, \dots, r$, are its Cartan generators acting in the j^{th} space, and the spectral parameter dependence has been omitted (see above). The expressions of type $K_1(\mathbf{q} + h^{(2)})$ are understood once one assumes that each auxiliary space \mathcal{V} is a diagonalizable module of \mathfrak{h} . In this case, $K_1(\mathbf{q} + h^{(2)})$ acts on a basis vector of $\mathcal{V}_1 \otimes \mathcal{V}_2$ as

$$K_1(\mathbf{q} + h^{(2)}) |v_1\rangle \otimes |v_2\rangle = \left(K_1(q_1 + \mu \lambda_1(v_2), \dots, q_r + \mu \lambda_r(v_2)) |v_1\rangle \right) \otimes |v_2\rangle \quad (1.3)$$

where $h_i |v_2\rangle = \lambda_i(v_2) |v_2\rangle$, $\forall i = 1, \dots, r$, and μ is an overall shift scale which shall be kept throughout the paper.

$K_1(\mathbf{q} + h^{(2)})$ can be rewritten consistently as

$$K_1(\mathbf{q} + h^{(2)}) = \exp \left(\sum_{i=1}^r \mu h_i^{(2)} \partial_{q_i} \right) K_1(\mathbf{q}) \exp \left(- \sum_{i=1}^r \mu h_i^{(2)} \partial_{q_i} \right) = e^{h^{(2)} \partial} K_1(\mathbf{q}) e^{-h^{(2)} \partial} \quad (1.4)$$

A_{12} is the dynamical Yang-Baxter R -matrix defining the bulk IRF model, and obeying the dynamical Yang-Baxter or Gervais-Neveu-Felder equation [6, 7]:

$$A_{12}(\mathbf{q} + h^{(3)}) A_{13}(\mathbf{q}) A_{23}(\mathbf{q} + h^{(1)}) = A_{23}(\mathbf{q}) A_{13}(\mathbf{q} + h^{(2)}) A_{12}(\mathbf{q}), \quad (1.5)$$

In the original case [5] one had $A_{12} = B_{21} = C_{12} = D_{21}$. Exchange algebra (1.2) can however be extended to a general situation with unrelated A , $B_{12} = C_{21}$ and D obeying a coupled set of dynamical Yang-Baxter equations together with suitable zero-weight and unitary relations. In this way, one gets a first dynamical extension of the quadratic exchange algebra (1.1) defined e.g. in [8, 9] as "boundary dynamical" of "fully dynamical" quadratic exchange algebra.

A second inequivalent dynamical extension arose [10] when studying the Poisson structure of the Lax formulation for the Ruijsenaars-Schneider model [11]. It reads

$$A_{12}(\mathbf{q}) K_1(\mathbf{q} + h^{(2)}) B_{12}(\mathbf{q}) K_2(\mathbf{q}) = K_2(\mathbf{q} + h^{(1)}) C_{12}(\mathbf{q}) K_1(\mathbf{q}) D_{12}(\mathbf{q}). \quad (1.6)$$

Studied in [12], it was shown in [13] to be a deformation by a Drinfeld twist of a non-dynamical bulk algebra $RTT = TTR$, following the ideas developed in [14–16], albeit with an associated non-trivial dynamical commuting quantum trace [17] (see also [18]).

We propose in this paper a third, a priori inequivalent and yet unknown, dynamical extension of the quantum exchange algebra. It reads

$$A_{12}(\mathbf{q}) K_1(\mathbf{q} - h^{(2)}) B_{12}(\mathbf{q}) K_2(\mathbf{q} + h^{(1)}) = K_2(\mathbf{q} - h^{(1)}) C_{12}(\mathbf{q}) K_1(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}). \quad (1.7)$$

Its particular, alternate-sign structure was suggested to us by preliminary studies [19] of the second Poisson structure of the Calogero-Moser rational model [20, 21].

We shall now describe the key features of this third dynamical "reflection" algebra. We first explain its occurrence from considerations on classical quadratic Poisson studies (section 2) and their associated dynamical Yang-Baxter equations. We then give in section 3 explicit realizations of its structure matrices and generating matrix K stemming from a bilinear formulation in terms of a quantum bulk T-matrix and its transposed. This allows a characterization of (1.7) as a general "dynamical twisted Yangian" structure. We establish a dynamical commuting quantum trace formula in section 4. We derive the fusion structures and dressing procedures of the dynamical reflection algebra in sections 5 and 6 respectively. We finally treat a simple example of dynamical reflection algebra, yielding explicit quantum integrable Hamiltonians through application of the trace construction..

2 From classical to quantum quadratic exchange algebra

Our proposition for a third quantum dynamical exchange algebra (generalized reflection algebra) follows from our observation of a particular form of the classical Yang-Baxter equations obeyed by some structure matrices in the description of the second Poisson structure of the rational Calogero-Moser model. It is therefore instructive to explain the connection between quantum dynamical reflection algebras and classical dynamical Yang-Baxter equations for general quadratic Poisson structures.

We shall from now on restrict ourselves to dynamical YB equations for which the defining Cartan algebra \mathfrak{h} is the Cartan algebra of A_{n-1} .

Let us first of all consider the classical structures. Given a classical Lax matrix ℓ , the most general quadratic form for the associated Poisson structure is

$$\{\ell_1, \ell_2\} = a_{12} \ell_1 \ell_2 + \ell_1 b_{12} \ell_2 - \ell_2 c_{12} \ell_1 - \ell_1 \ell_2 d_{12} \quad (2.1)$$

where consistency conditions imply that $a_{12} = -a_{21}$, $d_{12} = -d_{21}$, $b_{12} = c_{21}$. Note that (2.1) implies that the functions $\{\text{tr } \ell^m, m \in \mathbb{Z}_+\}$ Poisson-commute if $a + b = c + d$. A more general trace formula, $\text{tr } (\gamma^{-1} \ell)^m$, occurs whenever a scalar matrix γ exists such that

$$a_{12} \gamma_1 \gamma_2 + \gamma_1 b_{12} \gamma_2 - \gamma_2 c_{12} \gamma_1 - \gamma_1 \gamma_2 d_{12} = 0, \quad (2.2)$$

see [4].

Dynamical dependence of $abcd$ now is assumed to be on coordinates q_i , $i = 1, \dots, n$, on a dual \mathfrak{h}^* of the Cartan subalgebra \mathfrak{h} in $sl(n, \mathbb{C})$.

Associativity for the PB structure (2.1) is implied by algebraic consistency conditions (Yang-Baxter classical equations) for a, b, c, d , provided the a priori undetermined bracket $\{r_{12}, \ell_3\}$, $r = a, b, c, d$, be of an algebraic form. We consider here the following form for this PB

$$\{r_{12}, \ell_3\} = \epsilon_R h_3 \partial r_{12} \ell_3 + \epsilon_L \ell_3 h_3 \partial r_{12} \quad (2.3)$$

$$h \partial = \sum_{i=1}^n \mu e_{ii} \otimes \frac{\partial}{\partial q_i}, \quad (2.4)$$

where $e_{ii} \in \mathfrak{h}$, ϵ_R, ϵ_L are c-numbers to be later determined.

Jacobi identity for the PB is ensured by the following (sufficient) classical dynamical Yang-Baxter equations:

$$[a_{12}, a_{13}] + [a_{12}, a_{23}] + [a_{32}, a_{13}] + \epsilon_R (h_3 \partial a_{12} + h_1 \partial a_{23} + h_2 \partial a_{31}) = 0, \quad (2.5)$$

$$[d_{12}, d_{13}] + [d_{12}, d_{23}] + [d_{32}, d_{13}] + \epsilon_L (h_3 \partial d_{12} + h_1 \partial d_{23} + h_2 \partial d_{31}) = 0, \quad (2.6)$$

$$[a_{12}, c_{13} + c_{23}] + [c_{13}, c_{23}] - \epsilon_L h_3 \partial a_{12} - \epsilon_R h_1 \partial c_{23} + \epsilon_R h_2 \partial c_{13} = 0, \quad (2.7)$$

$$[d_{12}, b_{13} + b_{23}] + [b_{13}, b_{23}] - \epsilon_R h_3 \partial d_{12} + \epsilon_L h_1 \partial b_{23} - \epsilon_L h_2 \partial b_{13} = 0. \quad (2.8)$$

In absence of dynamical term, one recovers the usual classical quadratic algebra [4].

One then immediately observes that (2.5)-(2.8) is a classical limit ($\hbar \rightarrow 0$) of a set of 4 dynamical Yang-Baxter equations:

$$A_{12}(\mathbf{q}) A_{13}(\mathbf{q} - \epsilon_R h^{(2)}) A_{23}(\mathbf{q}) = A_{23}(\mathbf{q} - \epsilon_R h^{(1)}) A_{13}(\mathbf{q}) A_{12}(\mathbf{q} - \epsilon_R h^{(3)}), \quad (2.9)$$

$$D_{12}(\mathbf{q} + \epsilon_L h^{(3)}) D_{13}(\mathbf{q}) D_{23}(\mathbf{q} + \epsilon_L h^{(1)}) = D_{23}(\mathbf{q}) D_{13}(\mathbf{q} + \epsilon_L h^{(2)}) D_{12}(\mathbf{q}), \quad (2.10)$$

$$A_{12}(\mathbf{q}) C_{13}(\mathbf{q} - \epsilon_R h^{(2)}) C_{23}(\mathbf{q}) = C_{23}(\mathbf{q} - \epsilon_R h^{(1)}) C_{13}(\mathbf{q}) A_{12}(\mathbf{q} + \epsilon_L h^{(3)}), \quad (2.11)$$

$$D_{12}(\mathbf{q} - \epsilon_R h^{(3)}) B_{13}(\mathbf{q}) B_{23}(\mathbf{q} + \epsilon_L h^{(1)}) = B_{23}(\mathbf{q}) B_{13}(\mathbf{q} + \epsilon_L h^{(2)}) D_{12}(\mathbf{q}), \quad (2.12)$$

where the classical limit is defined by setting

$$R(\mathbf{q}) = \mathbb{I} + \hbar r(\mathbf{q}) + o(\hbar^2), \quad R = A, B, C, D \quad \text{and} \quad r = a, b, c, d \quad (2.13)$$

$$h^{(i)} = \hbar h_i + o(\hbar^3), \quad (2.14)$$

and keeping the order \hbar^2 in (2.9)-(2.12), orders 1 and \hbar being trivial.

These 4 equations are in turn characterized as sufficient conditions for associativity of a quantum quadratic dynamical exchange algebra:

$$A_{12}(\mathbf{q}) K_1(\mathbf{q} - \epsilon_R h^{(2)}) B_{12}(\mathbf{q}) K_2(\mathbf{q} + \epsilon_L h^{(1)}) = K_2(\mathbf{q} - \epsilon_R h^{(1)}) C_{12}(\mathbf{q}) K_1(\mathbf{q} + \epsilon_L h^{(2)}) D_{12}(\mathbf{q}) \quad (2.15)$$

assuming a set of zero-weight conditions

$$\epsilon_R [h^{(1)} + h^{(2)}, A_{12}] = \epsilon_L [h^{(1)} + h^{(2)}, D_{12}] = 0 \quad (2.16)$$

$$[\epsilon_R h^{(1)} - \epsilon_L h^{(2)}, C_{12}] = [\epsilon_L h^{(1)} - \epsilon_R h^{(2)}, B_{12}] = 0, \quad (2.17)$$

and unitary hypothesis

$$A_{12} A_{21} = D_{12} D_{21} = \mathbb{I} \otimes \mathbb{I} \quad ; \quad C_{12} = B_{21}. \quad (2.18)$$

Altogether, these relations ensure associativity of the product in the dynamical algebra.

Note that it is always possible to redefine the overall sign of the dynamical variable \mathbf{q} , which in turn leads to a global sign change of ϵ_R, ϵ_L . It follows that only the relative sign between ϵ_R and ϵ_L has relevance.

Note finally that the zero-weight conditions (2.17) put strong constraints on acceptable values of the ratio of the c-numbers ϵ_R, ϵ_L unless B_{12} and C_{12} belong to $\mathfrak{h} \otimes \mathfrak{h}$: they must be ratios of weights of the corresponding Lie algebra. We shall not discuss this issue any more at this time.

There exist in the literature two examples of the above algebra:

1. The dynamical boundary algebra [5] corresponds to $\epsilon_L = 1$ and $\epsilon_R = -1$. We are not aware at this stage of explicit classical examples for the realization of (2.3).

2. The semi-dynamical reflection algebra has $\epsilon_R = 0$ and $\epsilon_L = 1$. It is classically realized as (2.1) and (2.3) by the Lax representation of the rational Ruijsenaars-Schneider model [11].

Our proposition corresponds to $\epsilon_L = \epsilon_R = 1$. The dynamical Yang-Baxter equations now read:

$$A_{12}(\mathbf{q}) A_{13}(\mathbf{q} - h^{(2)}) A_{23}(\mathbf{q}) = A_{23}(\mathbf{q} - h^{(1)}) A_{13}(\mathbf{q}) A_{12}(\mathbf{q} - h^{(3)}), \quad (2.19)$$

$$D_{12}(\mathbf{q} + h^{(3)}) D_{13}(\mathbf{q}) D_{23}(\mathbf{q} + h^{(1)}) = D_{23}(\mathbf{q}) D_{13}(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}), \quad (2.20)$$

$$A_{12}(\mathbf{q}) C_{13}(\mathbf{q} - h^{(2)}) C_{23}(\mathbf{q}) = C_{23}(\mathbf{q} - h^{(1)}) C_{13}(\mathbf{q}) A_{12}(\mathbf{q} + h^{(3)}), \quad (2.21)$$

$$D_{12}(\mathbf{q} - h^{(3)}) B_{13}(\mathbf{q}) B_{23}(\mathbf{q} + h^{(1)}) = B_{23}(\mathbf{q}) B_{13}(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}). \quad (2.22)$$

It is at least partially realized (at least for (2.3)) by the 2×2 Lax and $abcd$ matrices describing the second Poisson structure of the Calogero-Moser model [19]. It is therefore justified to consider it for its own sake.

3 Explicit realizations: the twisted dynamical Yangian

We shall first of all establish the existence of this third dynamical reflection algebra and its associated dynamical Yang-Baxter equations by giving an explicit realization of these Yang-Baxter equations from any given dynamical R -matrix A obeying (2.9). We recall that from now on we fix $\epsilon_L = \epsilon_R = 1$.

Before proceeding, we must introduce notations and several technical lemmas which are essential to the manipulation of dynamical Yang-Baxter equations.

Definition 3.1 *Given a matrix*

$$M(\mathbf{q}) = \sum_{i,j=1}^n M_{ij}(\mathbf{q}) e_{ij} \in sl(n, \mathbb{C}), \quad (3.1)$$

one denotes by respectively M^{sl} and M^{sc} the shifted matrices

$$M^{sl}(\mathbf{q}) = \sum_{i,j=1}^n e^{\mu \partial_{q_i}} M_{ij}(\mathbf{q}) e^{-\mu \partial_{q_i}} e_{ij} \in sl(n, \mathbb{C}), \quad (3.2)$$

$$M^{sc}(\mathbf{q}) = \sum_{i,j=1}^n e^{\mu \partial_{q_j}} M_{ij}(\mathbf{q}) e^{-\mu \partial_{q_j}} e_{ij} \in sl(n, \mathbb{C}). \quad (3.3)$$

They are consistently defined as \mathbb{C} -number matrices with shifts on q_k coordinates in each matrix element $M_{ij}(\mathbf{q})$, resp. $q_k \rightarrow q_k + \mu \delta_{ik}$ and $q_k \rightarrow q_k + \mu \delta_{jk}$. In matrix form, the definition reads

$$M(\mathbf{q})^{sc} = \left(e^{h\partial} \left(M(\mathbf{q}) e^{-h\partial} \right)^t \right)^t \quad \text{and} \quad M(\mathbf{q})^{sl} = \left(\left(e^{h\partial} M(\mathbf{q}) \right)^t e^{-h\partial} \right)^t, \quad (3.4)$$

where $h\partial$ is defined in (2.4).

More generally, on a tensor product matrix one shall define shift $\mathbf{s} \mathbf{l}_a$ and $\mathbf{s} \mathbf{c}_a$ self-explanatorily.

Lemma 3.2 *If M_{12} is zero-weight under adjoint action of $\mathfrak{h} \otimes \mathfrak{h}$ as*

$$[h^{(1)} + h^{(2)}, M_{12}] = 0 \quad (3.5)$$

then the coadjoint action of $\exp \sum_j \mu (h_j^{(1)} \partial_{q_j} + h_j^{(2)} \partial_{q_j})$ on M_{12} yields a \mathbb{C} -number matrix

$$\widetilde{M}_{12} = M_{12}^{\mathbf{s} \mathbf{l}_1, \mathbf{s} \mathbf{c}_2} = M_{12}^{\mathbf{s} \mathbf{c}_1, \mathbf{s} \mathbf{l}_2}. \quad (3.6)$$

In other words, one has

$$e^{h^{(1)} \partial} M_{12}(\mathbf{q}) e^{-h^{(2)} \partial} = e^{-h^{(2)} \partial} \widetilde{M}_{12}(\mathbf{q}) e^{h^{(1)} \partial}. \quad (3.7)$$

The lemma is proven by direct calculation. We are now able to prove the following result

Theorem 3.3 *If $A_{12}(z_1, z_2; \mathbf{q})$ is unitary and obeys the dynamical Yang-Baxter equation (2.9) with zero-weight condition $[h^{(1)} + h^{(2)}, A_{12}] = 0$, we define the following matrices*

$$D_{12}(z_1, z_2; \mathbf{q}) = (A_{21}^{t_1 t_2}(f(z_2), f(z_1); \mathbf{q}))^{\mathbf{s} \mathbf{l}_1, \mathbf{s} \mathbf{l}_2}, \quad (3.8)$$

$$C_{12}(z_1, z_2; \mathbf{q}) = (A_{21}^{t_2}(f(z_2), z_1; \mathbf{q}))^{\mathbf{s} \mathbf{c}_2} \quad (3.9)$$

$$B_{12}(z_1, z_2; \mathbf{q}) = (A_{12}^{t_1}(f(z_1), z_2; \mathbf{q}))^{\mathbf{s} \mathbf{c}_1} = C_{21}(z_2, z_1; \mathbf{q}) \quad (3.10)$$

where $f(z)$ is any \mathbb{C} -valued function of the spectral parameter.

These matrices obey the Yang-Baxter equations (2.10)-(2.12) and $D_{12}(z_1, z_2; \mathbf{q})$ is unitary.

The proof is again by direct calculation, using relations of the type

$$(M e^{h \partial} \widehat{P})^t = (\widehat{P}^t)^{\mathbf{s} \mathbf{c}} e^{h \partial} (M^t)^{-\mathbf{s} \mathbf{l}}, \quad (3.11)$$

valid for any matrix M and \widehat{M} , and various partial transpositions of the Yang-Baxter equation for A .

Theorem 3.4 *We consider a representation of the dynamical Yang-Baxter equation for A by a quantum Lax matrix T in $\text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{H})$ for \mathcal{H} a (quantum) Hilbert space, assuming in addition a zero-weight condition for T under $\mathfrak{h} + \mathfrak{h}^{(q)}$. Here \mathcal{H} too is a diagonalizable module of \mathfrak{h} in order to be able to define $\mathfrak{h}^{(q)}$:*

$$A_{12}(z_1, z_2; \mathbf{q}) T_1(z_1; \mathbf{q} - h^{(2)}) T_2(z_2; \mathbf{q}) = T_2(z_2; \mathbf{q} - h^{(1)}) T_1(z_1; \mathbf{q}) A_{12}(z_1, z_2; \mathbf{q} - h^{(q)}) \quad (3.12)$$

$$[h^{(1)} + h^{(q)}, T_{1q}(z)] = 0, \quad (3.13)$$

where $h^{(q)}$ denotes the action of the Cartan algebra generators on the Hilbert space \mathcal{H} , that is assumed to be a diagonalizable module of the Cartan algebra.

We also define the "transposed" Lax matrix \mathcal{T} as

$$\mathcal{T}(z; \mathbf{q}) = \left(T^t(f(z); \mathbf{q}) \right)^{sc}, \quad (3.14)$$

where $f(z)$ is the same function as in Theorem 3.3. Transposition and shifts here act on the auxiliary space indices. It obeys a transposed exchange relation and a crossed exchange relation:

$$D_{12}(z_1, z_2; \mathbf{q} - h^{(q)}) \mathcal{T}_1(z_1; \mathbf{q}) \mathcal{T}_2(z_2; \mathbf{q} + h^{(1)}) = \mathcal{T}_2(z_2; \mathbf{q}) \mathcal{T}_1(z_1; \mathbf{q} + h^{(2)}) D_{12}(z_1, z_2; \mathbf{q}) \quad (3.15)$$

$$\mathcal{T}_1(z_1; \mathbf{q} - h^{(2)}) B_{12}(z_1, z_2; \mathbf{q}) \mathcal{T}_2(z_2; \mathbf{q} + h^{(1)}) = \mathcal{T}_2(z_2; \mathbf{q}) C_{21}(z_1, z_2; \mathbf{q} - h^{(q)}) \mathcal{T}_1(z_1; \mathbf{q}) \quad (3.16)$$

where the B, C, D matrices are given as in (3.10). If a \mathbb{C} -number matrix γ exists such that

$$A_{12}(z_1, z_2; \mathbf{q}) \gamma_1(z_1; \mathbf{q}) B_{12}(z_1, z_2; \mathbf{q}) \gamma_2(z_2; \mathbf{q}) = \gamma_2(z_2; \mathbf{q}) C_{12}(z_1, z_2; \mathbf{q}) \gamma_1(z_1; \mathbf{q}) D_{12}(z_1, z_2; \mathbf{q}) \quad (3.17)$$

then the operator valued matrix

$$K(z; \mathbf{q}) = T(z; \mathbf{q}) \gamma(z; \mathbf{q} + h^{(q)}) \mathcal{T}(z; \mathbf{q}) \quad (3.18)$$

realizes the dynamical quadratic algebra (1.7).

Again, the proof is done by direct calculation.

Examples of γ matrices for the dynamical twisted reflection algebra will be given in section 7, thereby establishing the existence of explicit realizations of the proposed third dynamical reflection algebra.

Remark 3.1 This construction can be characterized as a dynamical extension of the twisted (quantum) Yangian construction [22–24], in the specific case when the anti-automorphism σ is chosen to be the transposition.

Remark 3.2 A similar construction occurs for the boundary dynamical algebra ($\epsilon_L = 1 = -\epsilon_R$) except that σ is there chosen to be the inverse. Modifications on the definition of B, C, D from A imply that one has $A = C = B^{-1} = D^{-1}$. Hence, in the boundary dynamical algebra case, the existence of at least one matrix $\gamma = \mathbb{I}$ is trivially guaranteed.

In the generic $ABCD$ case, one proves the existence of two coactions:

Theorem 3.5 *Let us assume that K obeys the dynamical exchange algebra*

$$A_{12}(\mathbf{q}) K_1(\mathbf{q} - h^{(2)}) B_{12}(\mathbf{q}) K_2(\mathbf{q} + h^{(1)}) = K_2(\mathbf{q} - h^{(1)}) C_{12}(\mathbf{q}) K_1(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}) \quad (3.19)$$

where A, B, C, D obey (2.19)-(2.22). Introduce L and J obeying respectively the following exchange relations

$$A_{12}(\mathbf{q}) L_1(\mathbf{q} - h^{(2)}) L_2(\mathbf{q}) = L_2(\mathbf{q} - h^{(1)}) L_1(\mathbf{q}) A_{12}(\mathbf{q} + \alpha h^{(q)}) \quad (3.20)$$

$$D_{12}(\mathbf{q} + \alpha h^{(q)}) J_1(\mathbf{q}) J_2(\mathbf{q} + h^{(1)}) = J_2(\mathbf{q}) J_1(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}) \quad (3.21)$$

$$J_1(\mathbf{q} - h^{(2)}) B_{12}(\mathbf{q}) L_2(\mathbf{q} + h^{(1)}) = L_2(\mathbf{q}) B_{12}(\mathbf{q} + \alpha h^{(q)}) J_1(\mathbf{q}) \quad (3.22)$$

$$J_2(\mathbf{q} - h^{(1)}) C_{12}(\mathbf{q}) L_1(\mathbf{q} + h^{(2)}) = L_1(\mathbf{q}) C_{12}(\mathbf{q} + \alpha h^{(q)}) J_2(\mathbf{q}) \quad (3.23)$$

$$[\alpha h^{(q)} - h^{(1)}, L_1] = 0 \quad \text{and} \quad [\alpha h^{(q)} + h^{(1)}, J_1] = 0 \quad (3.24)$$

where q labels a Hilbert space \mathcal{H} carried by T and L and α is any \mathbb{C} -number. Then,

$$\tilde{K}(\mathbf{q}) = L(\mathbf{q}) K(\mathbf{q} + \alpha h^{(q)}) T(\mathbf{q}) \quad (3.25)$$

realizes also (3.19).

The proof follows again from a direct computation. In the reduced situation described by Theorem 3.3, an example of the construction described in Theorem 3.5 is precisely given by Theorem 3.4.

Example 1: $\alpha = 1$ Taking $J_1(z_1) = D_{1a}(z_1, 0)$ and $L_1(z_1) = C_{1a}(z_1, 0)$ realize (3.20)-(3.24). The quantum space \mathcal{H} is then identified with the auxiliary space \mathcal{V} , as it is standard in the construction of spin chain monodromy matrices.

Example 2: $\alpha = -1$ Taking $J_1(z_1) = B_{1a}(z_1, 0)$ and $L_1(z_1) = A_{1a}(z_1, 0)$ realize (3.20)-(3.24). Again, in the reduced situation described by Theorem 3.3, successive implementation of this coaction realizes precisely (3.25), with

$$T_1(z_1; \mathbf{q}) = \prod_{1 \leq i \leq n}^{\leftarrow} A_{1,a_i} \left(z_1, 0; \mathbf{q} - \sum_{j=i+1}^n h^{(a_j)} \right), \quad (3.26)$$

a well-known formula for the homogeneous dynamical bulk monodromy, see [7, 25].

These two examples can be combined to build realizations of the dynamical exchange algebra as dressing of an initial scalar solution γ (to be computed) by successive pairs (D, C) and (B, A) resp. on the right and left side of the $(n-1)$ sites monodromy matrix, together with consistent shifts on \mathbf{q} of the “internal” generators of this monodromy matrix according to (3.25).

4 Commuting traces

The general procedure to build a generating functional for commuting operators associated with the algebraic structure (3.19) follows from arguments formally similar to the cases of the other two dynamical reflection algebras. It is summarized in the following theorem

Theorem 4.1 *The following operators*

$$H_j = \text{Tr}_j \left(e^{\partial_j} K_j(z; \mathbf{q}) e^{\partial_j} K_j^+(z; \mathbf{q})^t \right) \quad (4.1)$$

commute with one another for every choice of pairs of distinct auxiliary spaces $\mathcal{V}_j, \mathcal{V}_k$. The trace Tr_j is only taken over the vector space indices whenever \mathcal{V}_j is a loop space $\mathcal{V}_j = V_j \otimes \mathbb{C}[[z_j]]$. The notation ∂_j is shorthand for

$$\partial_j = \sum_{k=1}^n \mu h_{kk}^{(j)} \frac{\partial}{\partial q_k}. \quad (4.2)$$

$K_{(j,k)}(z; \mathbf{q})$ obey the general exchange relation (3.19) for distinct auxiliary spaces $\mathcal{V}_j, \mathcal{V}_k$, and $K_{(j,k)}^+$ obey the dual dynamical reflection equation

$$\tilde{A}_{jk}(\mathbf{q}) K_j^+(\mathbf{q} - h^{(k)})^{sc_j} \tilde{B}_{jk}(\mathbf{q}) K_k^+(\mathbf{q} + h^{(j)})^{sc_k} = K_k^+(\mathbf{q} - h^{(j)})^{sc_k} \tilde{C}_{jk}(\mathbf{q}) K_j^+(\mathbf{q} + h^{(k)})^{sc_j} \tilde{D}_{jk}(\mathbf{q}) \quad (4.3)$$

where

$$\tilde{A}_{jk}(\mathbf{q}) = \left(A_{jk}^{-1}(\mathbf{q}) \right)^{t_j t_k} \quad ; \quad \tilde{B}_{jk}(\mathbf{q}) = \left((B_{jk}^{t_k}(\mathbf{q}))^{-1} \right)^{t_j} \quad (4.4)$$

$$\tilde{C}_{jk}(\mathbf{q}) = \left((C_{jk}^{t_j}(\mathbf{q}))^{-1} \right)^{t_k} = \tilde{B}_{kj}(z_k, z_j; \mathbf{q}) \quad ; \quad \tilde{D}_{jk}(\mathbf{q}) = \left(D_{jk}^{t_j t_k}(\mathbf{q}) \right)^{-1} \quad (4.5)$$

The proof of this theorem is a long and technical calculation that has been detailed in [26] for another type of dynamical reflection algebras. In our case, the proof follows the same lines, with appropriate change of signs.

Such a choice of distinct auxiliary spaces leading to non-trivial sets of commuting quantum Hamiltonians is available in at least two well-known situations:

First of all when the auxiliary spaces $\mathcal{V}_j, \mathcal{V}_k$ are isomorphic (but not identical) loop spaces, the restricted trace over vector indices yields a generating function $t(z)$ where z is the spectral parameter. The Theorem then establishes that $[t(z_1), t(z_2)] = 0$. Hence the operatorial coefficients of the formal series expansion of $t(z)$ in powers of z provide a set of mutually commuting Hamiltonians. This is the standard procedure in e.g. the case of quantum integrable spin chains.

The second case corresponds to the so-called quantum power traces such as originally described in [27]. It stems from the existence of a systematic procedure to construct successive tensorial powers of an initial finite dimensional auxiliary space together with the corresponding coefficient matrices $ABCD$. This procedure will be described presently in Sections 5 and 6 as “fusion” and “dressing”.

Let us now discuss more precisely the dual reflection equation. One can define a reduced representation of the coefficient matrices of the dual equation following Theorem 3.3:

Corollary 4.2 *If A, B, C, D obey the relations given in theorem 3.3*

$$D_{12}(z_1, z_2; \mathbf{q}) = \left(A_{21}^{t_1 t_2}(f(z_2), f(z_1); \mathbf{q}) \right)^{s_{l_1}, s_{l_2}}, \quad (4.6)$$

$$C_{12}(z_1, z_2; \mathbf{q}) = \left(A_{21}^{t_2}(f(z_2), z_1; \mathbf{q}) \right)^{s_{c_2}} \quad (4.7)$$

$$B_{12}(z_1, z_2; \mathbf{q}) = \left(A_{12}^{t_1}(f(z_1), z_2; \mathbf{q}) \right)^{s_{c_1}} = C_{21}(z_2, z_1; \mathbf{q}) \quad (4.8)$$

then the same relations are valid for $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ given in theorem 4.1:

$$\tilde{D}_{12}(z_1, z_2; \mathbf{q}) = \left(\tilde{A}_{21}^{t_1 t_2}(f(z_2), f(z_1); \mathbf{q}) \right)^{s_{l_1}, s_{l_2}}, \quad (4.9)$$

$$\tilde{C}_{12}(z_1, z_2; \mathbf{q}) = \left(\tilde{A}_{21}^{t_2}(f(z_2), z_1; \mathbf{q}) \right)^{s_{c_2}} \quad (4.10)$$

$$\tilde{B}_{12}(z_1, z_2; \mathbf{q}) = \left(\tilde{A}_{12}^{t_1}(f(z_1), z_2; \mathbf{q}) \right)^{s_{c_1}} \quad (4.11)$$

In addition, this reduced representation induces a relation between direct and dual scalar solutions as follows

Corollary 4.3 *If A, B, C, D obey the relations given in theorem 3.3 with $f(z) = -z$, and if moreover A obeys the crossing relation*

$$\left(A_{12}^{t_1}(z_1, z_2; \mathbf{q}) \right)^{-1} = \left(A_{12}^{-1}\left(z_1 + \frac{\eta}{2}, z_2 - \frac{\eta}{2}; \mathbf{q}\right) \right)^{t_2} \quad (4.12)$$

then, from any solution $\mathbb{K}(z; \mathbf{q})$ to the dynamical reflection equation (3.19), one can construct a solution $K^+(z; \mathbf{q})$ to the dual dynamical reflection equation (4.3) as

$$K^+(z; \mathbf{q}) = \left(\left(\mathbb{K}^t\left(z + \frac{\eta}{2}; \mathbf{q}\right) \right)^{-1} \right)^{-s_c} \quad (4.13)$$

In that case, the expression of Hamiltonian (4.1) simplifies to

$$H_j = \text{Tr}_j \left(e^{\partial_j} K_j(z; \mathbf{q}) \mathbb{K}_j(z; \mathbf{q})^{-1} e^{\partial_j} \right) \quad (4.14)$$

with now K_j and \mathbb{K}_j solutions to the same dynamical reflection equation (3.19).

5 Fusion procedures

The coaction procedure in Section 3 described tensoring of "quantum" spaces. We now turn to the fusion procedure, that allows to construct higher spin representations of the quantum algebra by a consistent tensoring of auxiliary spaces. This "auxiliary" tensoring also play a key role (as commented before) in defining higher powers in quantum traces of monodromy matrices, realizing through application of the quantum trace formula the quantum analogue of the classical $\text{Tr}(L^n)$ for a Lax matrix (see [17, 25, 27]).

We shall restrict ourselves to the first step, i.e. definition of a consistent tensor product on auxiliary spaces. The next step, i.e. projection on irreducible representations, is a complex and delicate issue which should be examined separately.

The tensoring procedure actually follows from two fundamental lemmas, proved by direct computation using suitable Yang-Baxter equations:

Lemma 5.1 *Given two representations of the dynamical exchange algebra, respectively:*

$$A_{12}(\mathbf{q}) K_1(\mathbf{q} - h^{(2)}) B_{12}(\mathbf{q}) K_2(\mathbf{q} + h^{(1)}) = K_2(\mathbf{q} - h^{(1)}) C_{12}(\mathbf{q}) K_1(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}) \quad (5.1)$$

$$A_{12}(\mathbf{q}) K'_1(\mathbf{q} - h^{(2)}) B_{12}(\mathbf{q}) K'_2(\mathbf{q} + h^{(1)}) = K'_2(\mathbf{q} - h^{(1)}) C_{12}(\mathbf{q}) K'_1(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}) \quad (5.2)$$

with the same quantum space \mathcal{H} , the following K -matrices and structural matrices:

$$A_{<11'>2}(\mathbf{q}) = A_{1'2}(\mathbf{q} - h^{(1)}) A_{12}(\mathbf{q}) \quad ; \quad D_{<11'>2}(\mathbf{q}) = D_{1'2}(\mathbf{q}) D_{12}(\mathbf{q} + h^{(1')}) \quad ; \quad (5.3)$$

$$B_{<11'>2}(\mathbf{q}) = B_{1'2}(\mathbf{q}) B_{12}(\mathbf{q} + h^{(1')}) \quad ; \quad C_{<11'>2}(\mathbf{q}) = C_{1'2}(\mathbf{q} - h^{(1)}) C_{12}(\mathbf{q}) \quad ; \quad (5.4)$$

$$h^{(<11'>)} = h^{(1)} + h^{(1')} \quad \text{and} \quad K_{<11'>}(\mathbf{q}) = K'_{1'}(\mathbf{q} - h^{(1)}) B_{11'}(\mathbf{q}), K_1(\mathbf{q} + h^{(1')}) \quad (5.5)$$

obey the dynamical exchange algebra:

$$\begin{aligned} A_{<11'>2}(\mathbf{q}) K_{<11'>}(\mathbf{q} - h^{(2)}) B_{<11'>2}(\mathbf{q}) K_2(\mathbf{q} + h^{(<11'>)}) = \\ = K_2(\mathbf{q} - h^{(<11'>)}) C_{<11'>2}(\mathbf{q}) K_{<11'>}(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}) \end{aligned} \quad (5.6)$$

Lemma 5.2 *Given two representations of the dynamical exchange algebra, respectively:*

$$A_{12}(\mathbf{q}) K_1(\mathbf{q} - h^{(2)}) B_{12}(\mathbf{q}) K_2(\mathbf{q} + h^{(1)}) = K_2(\mathbf{q} - h^{(1)}) C_{12}(\mathbf{q}) K_1(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}) \quad (5.7)$$

$$A_{12}(\mathbf{q}) K'_1(\mathbf{q} - h^{(2)}) B_{12}(\mathbf{q}) K'_2(\mathbf{q} + h^{(1)}) = K'_2(\mathbf{q} - h^{(1)}) C_{12}(\mathbf{q}) K'_1(\mathbf{q} + h^{(2)}) D_{12}(\mathbf{q}) \quad (5.8)$$

with the same quantum space \mathcal{H} , the following K -matrices and structural matrices:

$$A_{1<22'>}(\mathbf{q}) = A_{12}(\mathbf{q}) A_{12'}(\mathbf{q} - h^{(2)}) \quad ; \quad D_{1<22'>}(\mathbf{q}) = D_{12}(\mathbf{q} + h^{(2')}) D_{12'}(\mathbf{q}) \quad ; \quad (5.9)$$

$$B_{1<22'>}(\mathbf{q}) = B_{12'}(\mathbf{q} - h^{(2')}) B_{12}(\mathbf{q}) \quad ; \quad C_{1<22'>}(\mathbf{q}) = C_{12'}(\mathbf{q}) C_{12}(\mathbf{q} - h^{(2')}) \quad ; \quad (5.10)$$

$$h^{(<22'>)} = h^{(2)} + h^{(2')} \quad \text{and} \quad K_{<22'>}(\mathbf{q}) = K'_{2'}(\mathbf{q} - h^{(2)}) B_{2'2}(\mathbf{q}), K_2(\mathbf{q} + h^{(2')}) \quad (5.11)$$

obey the dynamical exchange algebra:

$$\begin{aligned} A_{1<22'>}(\mathbf{q}) K_1(\mathbf{q} - h^{(<22'>)}) B_{1<22'>}(\mathbf{q}) K_{<22'>}(\mathbf{q} + h^{(1)}) = \\ = K_{<22'>}(\mathbf{q} - h^{(1)}) C_{1<22'>}(\mathbf{q}) K_1(\mathbf{q} + h^{(<22'>)}) D_{1<22'>}(\mathbf{q}) \end{aligned} \quad (5.12)$$

Note that if $\mathcal{V}_1 = \mathcal{V}_2$ and $\mathcal{V}_{1'} = \mathcal{V}_{2'}$, one can show that if one assumes the unitary relations $C_{12} = B_{21}$, $A_{12} A_{21} = \mathbb{I} = D_{12} D_{21}$ and zero-weight condition for A and D , one has

$$A_{<11'><22'>} A_{<22'><11'>} = \mathbb{I} = D_{<11'><22'>} D_{<22'><11'>} , \quad (5.13)$$

$$C_{<11'><22'>} = B_{<22'><11'>} . \quad (5.14)$$

Hence, the unitary condition is indeed preserved whenever relevant.

Successive iterations of both Lemmae allow to define multiply fused coefficient matrices $A_{\underline{M}\underline{N}}, B_{\underline{M}\underline{N}}, C_{\underline{M}\underline{N}}, D_{\underline{M}\underline{N}}$ and operator matrices $K_{\underline{M}}$ labeled by ordered sets of auxiliary space indices $\underline{M} = \langle 12 \dots m \rangle$, $\underline{N} = \langle 1'2' \dots n' \rangle$. Precisely, the fusion formulae are to be understood as describing the procedure of addition of a single space index, resp. $1'$ to an already fused multiple space index 1 (left-hand fusion) and $2'$ to an already fused multiple space index 2 (right-hand fusion). The final formulae are quite cumbersome but closely match the formulae given in Section IV-A of [17] with suitable changes in the sign shifts.

It is important to remark here that the Yang-Baxter equations guarantee that the fused coefficient matrices are univocally defined whichever order of implementation is defined to add left and right indices to resp. \underline{M} and \underline{N} . This is not surprising since Yang-Baxter equations are originally a guarantee of invariance under permutation of space indices in exchange processes.

6 Dressing procedures

As in the previous cases of (dynamical) boundary algebras, there exists a supplementary possibility to implement the dynamical quadratic algebra on a tensor product of auxiliary spaces by the so-called "dressing" procedure. Precisely, one has

Lemma 6.1 *If K obeys the general exchange relation (3.19), then the operator $Q K S$ also realizes this algebra, provided the operators Q and S verify the following relations*

$$A_{\underline{N}\underline{M}}(\mathbf{q}) Q_{\underline{N}}(\mathbf{q} - h^{(\underline{M})}) = Q_{\underline{N}}(\mathbf{q}) A_{\underline{N}\underline{M}}(\mathbf{q}) \quad \text{with} \quad h^{(\underline{M})} = h^{(1)} + h^{(2)} + \dots + h^{(m)} \quad (6.1)$$

$$C_{\underline{N}\underline{M}}(\mathbf{q}) Q_{\underline{N}}(\mathbf{q} + h^{(\underline{M})}) = Q_{\underline{N}}(\mathbf{q}) C_{\underline{N}\underline{M}}(\mathbf{q}) \quad (6.2)$$

$$D_{\underline{N}\underline{M}}(\mathbf{q}) S_{\underline{N}}(\mathbf{q}) = S_{\underline{N}}(\mathbf{q} + h^{(\underline{M})}) D_{\underline{N}\underline{M}}(\mathbf{q}) \quad (6.3)$$

$$B_{\underline{N}\underline{M}}(\mathbf{q}) S_{\underline{N}}(\mathbf{q}) = S_{\underline{N}}(\mathbf{q} - h^{(\underline{M})}) B_{\underline{N}\underline{M}}(\mathbf{q}) \quad (6.4)$$

$$[h_{\underline{N}}, Q_{\underline{N}}] = 0 = [h_{\underline{N}}, S_{\underline{N}}] \quad (6.5)$$

It is to be understood in this Lemma that the auxiliary space indices $\underline{M}, \underline{N}, \dots$ of the coefficient matrices are generically sets of fused indices obtained by implementation of the fusion Lemmae 5.1 and 5.2.

An explicit realization of Q and S is obtained as follows:

Lemma 6.2 *The following objects*

$$Q_{\underline{N}} \equiv Q_{1\dots n}(\mathbf{q}) = \overset{\vee}{A}_{21}(\mathbf{q}) \overset{\vee}{A}_{32}(\mathbf{q} + h^{(1)}) \dots \overset{\vee}{A}_{n,n-1}(\mathbf{q} + h^{(1)} + \dots + h^{(n-2)}) \quad (6.6)$$

$$S_{\underline{N}} \equiv S_{1\dots n}(\mathbf{q}) = \overset{\vee}{D}_{21}(\mathbf{q} + h^{(3)} + \dots + h^{(n)}) \overset{\vee}{D}_{32}(\mathbf{q} + h^{(4)} + \dots + h^{(n)}) \dots \overset{\vee}{D}_{n,n-1}(\mathbf{q}) \quad (6.7)$$

realize relations (6.1)-(6.5). We denote here $\overset{\vee}{R}_{12} = \mathcal{P}_{12} R_{12}$, $\forall R$, where \mathcal{P}_{ij} is the permutation operator acting on the auxiliary spaces $\mathcal{V}_i \otimes \mathcal{V}_j$.

Remark 6.1 The dressing procedure yields commuting traces that are completely different from the ones obtained from the fusion procedure, as described in section 5. In fact, one can show (at least in some particular cases) that the latter leads to completely factorized traces $\left(\text{Tr}(K e^{h\partial} \tilde{K} e^{h\partial})\right)^n$, while the former is closer to a form $\text{Tr}\left((K e^{h\partial} \tilde{K} e^{h\partial})^n\right)$. The dressing is useful for constructing an adequate set of independent quantum quantities, while fusion is more appropriate to build a local Hamiltonian. This key point of the dressing procedure was already pointed out in [17, 25, 27].

Remark 6.2 The permutation operator \mathcal{P}_{12} acts of course as $\mathcal{P}_{12} R_{12} \mathcal{P}_{12} = R_{21}$. If R_{12} does not depend on spectral parameters (i.e. the auxiliary spaces $V_1 = V_2$ are isomorphic to finite dimensional diagonalizable modules of the Cartan algebra \mathfrak{h}), the operator P_{12} is easily constructed as $P = \sum_{i,j=1}^n E_{ij} \otimes E_{ji}$ where E_{ij} are the elementary matrices of $\text{End}(V_1)$.

If however R_{12} depends on spectral parameters z_1, z_2 (i.e. the auxiliary spaces $\mathcal{V}_1, \mathcal{V}_2$ are loop spaces $V \otimes \mathbb{C}[[z_1]]$, $V \otimes \mathbb{C}[[z_2]]$), explicit implementation of the spectral parameter exchange as $\mathcal{P}_{12} f(z_1, z_2) \mathcal{P}_{12} = f(z_2, z_1)$ is not so easily available. It may then be difficult to explicitly define the quantum commuting operators analogous to $\text{Tr}\left((K e^{h\partial} \tilde{K} e^{h\partial})^n\right)$. Fortunately, expansion in formal powers of the spectral parameter z of the single, well-defined operator $\text{Tr}\left(K(z) e^{h\partial} \tilde{K}(z) e^{h\partial}\right)$ is in this case available. It provides an alternative viable procedure to obtain a family of algebraically independent quantum commuting operators.

7 Examples

We consider the following A, B, C, D matrices acting in $\mathbb{C}^n \otimes \mathbb{C}^n$:

$$A_{12} = \mathbb{I}_n \otimes \mathbb{I}_n + \sum_{1 \leq i \neq j \leq n} \frac{\mu}{q_i - q_j} \left(E_{ij} \otimes E_{ji} - E_{ii} \otimes E_{jj} \right) \quad (7.1)$$

$$B_{12} = \mathbb{I}_n \otimes \mathbb{I}_n + \sum_{1 \leq i \neq j \leq n} \frac{\mu}{q_i - q_j + \mu} \left(E_{ji} \otimes E_{ji} - E_{ii} \otimes E_{jj} \right) \quad (7.2)$$

$$C_{12} = \mathbb{I}_n \otimes \mathbb{I}_n + \sum_{1 \leq i \neq j \leq n} \frac{\mu}{q_i - q_j + \mu} \left(E_{ji} \otimes E_{ji} - E_{jj} \otimes E_{ii} \right) \quad (7.3)$$

$$D_{12} = \mathbb{I}_n \otimes \mathbb{I}_n + \sum_{1 \leq i \neq j \leq n} \frac{\mu}{q_i - q_j} \left(E_{ij} \otimes E_{ji} - E_{ii} \otimes E_{jj} \right) \quad (7.4)$$

where E_{ij} is the $n \times n$ elementary matrix with 1 at position (i, j) and 0 elsewhere. Those matrices are of the type given in Theorem 3.3.

Direct computations show that the following matrices

$$\gamma = \sum_{1 \leq i, j \leq n} m_i m_j E_{ij}, \quad m_i \in \mathbb{C} \quad (7.5)$$

$$\gamma = \sum_{1 \leq i, j \leq n} (q_i - q_j) m_i m_j E_{ij}, \quad m_i \in \mathbb{C} \quad (7.6)$$

$$\gamma = \sum_{i=1}^n \frac{f(\mathbf{q} + \mu e_{ii})}{f(\mathbf{q} - \mu e_{ii})} \prod_{k \neq i} (q_i - q_k) E_{ii}, \quad (7.7)$$

where f is an arbitrary \mathbb{C} -valued function and $e_{ii} \in \mathfrak{h}$, obey the dynamical reflection equation. Let us note that the two first solutions are not invertible.

Then, expression (4.14) with K_j of the form (7.5) and \mathbb{K}_j of the form (7.7) leads to the Hamiltonian

$$H = \sum_{\ell=1}^n \frac{\mu m_\ell^2}{\prod_{k=1}^n (q_\ell - q_k + \mu)} e^{2\mu \partial_{q_\ell}}. \quad (7.8)$$

The f -dependent factor in (7.7) is easily seen to be reabsorbed by a conjugation of H by $f(\mathbf{q})$. It has been therefore set to 1 in (7.8). From our construction of sections 4, 5 and 6, these Hamiltonians are in principle quantum integrable. We remark that they take a form close to the Ruijsenaars-Schneider Hamiltonians, although the coinciding points singularities have been replaced by finite distance singularities.

For the particular case $n = 2$, one gets

$$H = \left(\frac{m_1^2}{q + \mu} e^{2\mu \partial_q} - \frac{m_2^2}{q - \mu} e^{-2\mu \partial_q} \right) e^{2\mu \partial_Q} \quad (7.9)$$

where we have introduced the relative and center-of-mass coordinates $q = q_1 - q_2$ and $Q = q_1 + q_2$. The relative Hamiltonian has in particular eigenfunctions with zero eigenvalue taking the form

$$\begin{aligned} \psi_k(q) &= \frac{\Gamma(\frac{q+\mu}{4\mu} + \frac{1}{2})}{\Gamma(\frac{q+\mu}{4\mu})} e^{-\frac{q}{4\mu} \ln(\frac{m_1}{m_2})} \sin(k \frac{\pi q}{\mu}) \\ \varphi_k(q) &= \frac{\Gamma(\frac{q+\mu}{4\mu} + \frac{1}{2})}{\Gamma(\frac{q+\mu}{4\mu})} e^{-\frac{q}{4\mu} \ln(\frac{m_1}{m_2})} \cos(k \frac{\pi q}{\mu}) \end{aligned} \quad k \in \mathbb{Z}_{\geq 0}. \quad (7.10)$$

Acknowledgements

This work was sponsored by CNRS, Université de Cergy-Pontoise, Université de Savoie and ANR Project DIADEMS (Programme Blanc ANR SIMI1 2010-BLAN-0120-02). J.A. wishes to thank LAPTh for their kind hospitality.

References

- [1] I.V. Cherednik, *Factorizing particles on a half line and root systems*, Theor. Math. Phys. **61** (1984) 977.

- [2] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. **A21** (1988) 2375.
- [3] P.P. Kulish and E.K. Sklyanin, *Algebraic structures related to reflection equations*, J. Phys. **A25** (1992) 5963.
- [4] L. Freidel and J.M. Maillet, *Quadratic algebras and integrable systems*, Phys. Lett. **B262** (1991) 278.
- [5] R. E. Behrend, P. A. Pearce and D. O'Brien, *A construction of solutions to reflection equations for interaction-round-a-face models*, J. Stat. Phys. **84** (1996) 1. [arXiv:hep-th/9507118](#).
- [6] J.L. Gervais, A. Neveu, *Novel triangle relation and absence of tachyons in Liouville string field theory*, Nucl. Phys. **B238** (1984) 125.
- [7] G. Felder, *Elliptic quantum groups*, Proc. ICMP Paris 1994, pp 211 and [arXiv:hep-th/9412207](#).
- [8] H. Fan, B.-Y. Hou, G.-L. Li, and K.A. Shi, *Integrable $A_{n-1}^{(1)}$ IRF model with reflecting boundary condition*, Mod. Phys. Lett. **A26** (1997) 1929.
- [9] Z. Nagy and J. Avan, *Spin chains from dynamical quadratic algebras*, J. Stat. Mech. **2** (2005) P03005 and [arXiv:math/0501029](#).
- [10] G.E. Arutyunov, L.O. Chekhov and S.A. Frolov, Comm. Math. Phys **192** (1998) 405-432, [arXiv:q-alg/9612032](#);
G.E. Arutyunov and S.A. Frolov, Comm. Math. Phys. **191** (1998) 15-29 and [arXiv:q-alg/9610009](#).
- [11] S.N.M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Ann. Phys. **170** (1986) 370.
- [12] Z. Nagy, J. Avan and G. Rollet, *Construction of dynamical quadratic algebras*, Lett. Math. Phys. **67** (2004) 1-11 and [arXiv:math/0307026](#).
- [13] J. Avan and G. Rollet, *Parametrization of semi-dynamical quantum reflection algebra*, J. Phys. **A40** (2007) 2709-2731 and [arXiv:math/0611184](#).
- [14] M. Jimbo, H. Konno, S. Odake, J. Shiraishi, *Quasi-Hopf twistors for elliptic quantum groups*, "Transformation Groups" and [arXiv:q-alg/9712029](#).
- [15] D. Arnaudon, E. Buffenoir, E. Ragoucy, Ph. Roche, *Universal solutions of quantum dynamical Yang-Baxter equations*, Lett. Math. Phys. **44** (1998) 201 and [arXiv:q-alg/9712037](#).

- [16] E. Buffenoir, P. Roche and V. Terras, *Universal Vertex-IRF Transformation for Quantum Affine Algebras*, [arXiv:0707.0955](#);
Quantum Dynamical coBoundary Equation for finite dimensional simple Lie algebras, [arXiv:math/0512500](#).
- [17] Z. Nagy, J. Avan, A. Doikou and G. Rollet, *Commuting quantum traces for quadratic algebras*, J. Math. Phys. **46** (2005) 083516 and [arXiv:math/0403246](#).
- [18] J. Avan, C. Zambon *On the semi-dynamical reflection equation: solutions and structure matrices*, J. Phys. **A41** (2008) 194001 and [arXiv:0707.3036](#).
- [19] J. Avan and E. Ragoucy, work in progress.
- [20] C. Bartocci, G. Falqui, I. Mencattini, G. Ortenzi, M. Pedroni *On the geometric origin of the bi-Hamiltonian structure of the Calogero-Moser system* and [arXiv:0902.0953v2](#).
- [21] F. Magri, P. Casati, G. Falqui, M. Pedroni, *Eight lectures on Integrable Systems*, In: Integrability of Nonlinear Systems (Y. Kosmann-Schwarzbach et al. eds.), Lecture Notes in Physics 495 (2nd edition), 2004, pp. 209-250.
- [22] G. Olshanski, *Twisted Yangians and infinite-dimensional classical Lie algebras*, in ‘Quantum Groups’ (P. P. Kulish, Ed.), Lect. Notes in Math. **1510**, Springer, Berlin-Heidelberg, 1992, pp. 103–120.
- [23] A. Molev, M. Nazarov and G. Olshanski, *Yangians and classical Lie algebras*, Russian Math. Survey **51** (1996) 205 and [arXiv:hep-th/9409025](#).
- [24] A. Molev, E. Ragoucy and P. Sorba, *Coideal subalgebras in quantum affine algebras*, Rev. Math. Phys. **15** (2003) 789 and [arXiv:math.QA/0208140](#).
- [25] J. Avan, O. Babelon, E. Billey, *The Gervais–Neveu–Felder equation and quantum Calogero–Moser systems*, Comm. Math. Phys. **178** (1996) 281.
- [26] Z. Nagy, *Systèmes intégrables et algèbres de réflexion dynamiques*, PhD Thesis, Cergy Pontoise Univ. (2005), <http://biblioweb.u-cergy.fr/theses/05CERG0270.pdf>
- [27] J.-M. Maillet, *Lax equations and quantum groups*, Phys. Lett. **B245** (1990) 480.